SCHOOL MATHEMATICS -
WRITTEN WORK AND THE SPOKEN WORD:
TWO WINDOWS ON THE MIND

J. Dale Burnett

Queen's University, Kingston, Ontario

Paper presented at the Eighth PME Conference of the International Group for the Psychology of Mathematics Education, Sydney Australia, August 16 - 19, 1984.

SCHOOL MATHEMATICS -
WRITTEN WORK AND THE SPOKEN WORD:
TWO WINDOWS ON THE MIND

J. Dale Burnett

Queen's University, Kingston, Ontario

> In my Father's house are many mansions: if it were not so, I would have told you.
>
> St. John 14: 2

"Why am I doing this ?", in this particular instance writing a paper for the Eighth PME Conference. Activities that require a substantive commitment of time and energy should have at least partial, if incomplete, answers to questions such as this. Occasionally it is refreshing, and illuminating, to reflect on one's activities. This point, particularly as it pertains to professional practice, is comprehensively and eloquently discussed in a recent book by Schon (1983). The first few paragraphs of this paper are a personal attempt to reflect on the practice of scholarly writing. Additional reflective comments and questions are intended to act as a mortar for the article.

One might respond to the original question with, "It helps strengthen my curriculum-vitae to have references citing papers delivered at international conferences.". This is usually true. However the reply is superficial - noting only the article's existence and not it's substance. A more appropriate reply - more appropriate because it is fundamentally (in my opinion) more accurate - is to state that this article (as well as the others presented here) is an exercise in faith. In each case the author believes that he/she has something important and worthwhile to say and to share. Something that will make a contribution to our understanding of a particular phenomena or class of phenomena, and that this in turn will constitute another brick in the buildings of man.

Pursuing the metaphor, we all are aware that bricks come in different sizes, are made out of different materials and serve different functions, depending on where they are placed. Sometimes old bricks need to be removed to make way for newer bricks. Presentation in a public forum also constitutes an implicit invitation for feedback - how can we improve our bricks? This article is a brick - one contributing (hopefully) to the development of one small house - a theory of cognitive development. Even small buildings have rooms - one for methodology, one for assumptions, one for task domains, one for standards of description, and so forth. This article hopes to explore some of these rooms.

Switching metaphors (and underlying assumptions), one may contrast cognitive development to a roll of movie film. In this context three questions come immediately to mind:

(1) How do we select the subject for our camera?

(2) How do we ensure the picture is in focus?

(3) How does the view change over time?

The first two questions seem to be natural precursors to the third, and it is these two questions that motivated the present enquiry. Yet answers to the third question are the eventual goal, since these will form the basis for theories of cognitive development.

HOW DO WE SELECT THE SUBJECT FOR OUR CAMERA ?

What do we wish to observe?

Much has been written on problem solving. Yet often the more critical issue is that of problem selection, particularly for the educational researcher. The issue has two principal facets: what events do we wish to observe and who do we wish to observe? But each facet requires the consideration of many variables. Imagine for a moment the question of selecting or designing a suitable event for detailed investigation. Davis (1984) is one of the more recent authors to state, "...it is not surprising to find cognitive scientists turning often to the study of mathematical thought in order to learn about human thought in general...(p.2)". For many purposes mathematics has the advantage of being considered a relatively well-defined body of knowledge. In this sense it provides a clear standard or template for comparison with student responses. It also provides a template for comparison with so-called expert performance. We have the potentially rich situation of a triangle of perspectives: (1) established knowledge, (2) novice performance and (3) expert performance, although this study is limited to the first two perspectives. It would be interesting to observe both mathematics teachers and mathematicians while trying to solve the same problems used in this study.

Within mathematics there is still a myriad of possible topics to consider. One criterion that has often been used in the past, particularly where the primary concern has been to study problem solving, is to select a rather esoteric topic thereby attempting to control for prior knowledge. On the other hand, there is growing recognition in the value of looking at typical school-based tasks. This is essentially the issue of external validity - the degree to which observed effects can be generalized to other settings and populations. The present investigation took this latter perspective - the tasks to be studied are taken directly from the school curriculum.

Even within the school mathematics curriculum there are still many possibilities. One of the primary purposes of this study is to provide information, not only on student performance, but on the efficacy of different data collecting paradigms. The heart of the present investigation consists of a comparison of (1) a semi-structured interview on current maths topics with (2) a pre-set exam, each question followed by a short debriefing session.

<u>Who do we wish to observe?</u>

This report is the second in a planned series that provides a detailed account of students working with school or curriculum centered materials. The first study (Burnett, 1981) examined protocols of two Canadian students attempting to solve six mathematically isomorphic tasks (two simultaneous equations in two unknowns). The present study examines 12 Australian students from grade 11 under two different types of experimental setting. Each situation contained two students rated low in mathematical ability, two rated average and two rated high as judged by the classroom teacher. Approximately the same number of boys and girls were placed in each experimental cell.

Experimental Design

One setting consisted of a semi-structured interview where the student first indicated what topics were currently being discussed in class. The interviewer followed this with a series of questions intended to determine the nature of the student's understanding of these topics. The second format was more structured - each student was initially given a set of problems to solve with no interruption from the interviewer. Upon completion of each problem, the student was then asked to go over it again and describe what had been done. During this phase of the session the student was also asked a series of probing questions to clarify the rationale underlying the student's actions. The two task formats were designed to provide a comparison of the richness of the data obtained using each approach.

The structured setting for grade 11 involved a total of 14 written problems. Three of these problems were taken from the previous Canadian study and thus deal with two simultaneous equations. The remaining 11 problems were taken from the Mathematics Item Bank prepared by the Australian Council for Educational Research (1978).

HOW DO WE ENSURE THE PICTURE IS IN FOCUS ?

Assuming we are looking at something worthwhile, how do we ensure that we portray an accurate view of the phenomena? This is essentially the issue of internal validity - do we have an appropriate description? This is closely tied to the issue of analysis. Once again, there is a strong psychometric tradition in mathematics education that emphasizes the sophisticated statistical treatment of large collections of test scores. Krutetskii (1976) presents one of the better known statements contrasting this approach with that of protocol analysis:

> It is hard to understand how theory or practice can be enriched by [one] who computed, for 130 mathematically gifted adolescents, their scores on different kinds of tests and studied the correlation between them, finding in some cases it was significant and in others not. The process of solution did not interest the investigator. But what rich material could be provided by a study of the process of mathematical thinking in 130 mathematically able adolescents! (p. 14)

Newell and Simon (1972) provide one of the most thorough examples of such a protocol approach, although their tasks focused on chess, symbolic logic and cryptarithmetic. Ericsson and Simon (1984) have just published a comprehensive treatment of protocol analysis that is likely to become a major reference for researchers in the coming decade. This paper also reflects a protocol perspective. The remainder of the article is an attempt to portray events under the different experimental conditions described earlier. In the interests of continuity, comments on particular episodes will be inserted near the description of that episode. These comments will then be collected together in the last section of the paper, where we can reflect on what was gained by conducting the study.

Structured Setting - Set Problems

Sample: High Ability - 2 boys

Average Ability - 1 boy, 1 girl

Low Ability - 1 boy, 1 girl.

Problem 1

> Jane bought $1.30 worth of 10-cent and 12-cent stamps at the Post Office. If she bought 12 stamps, how many of each did she buy ?

This first problem is deliberately designed to give no indication that either algebra or "2 equations in 2 unknowns" is an appropriate frame of reference. It turns out that not one of the six students used a "school mathematics" approach. Both of the high ability students used a method of estimation and confirmation. In both cases the written work provides little indication of the strategy employed and would likely receive very low marks according to any conventional grading system.

12x	12	12x	10x
10	12	5	7
1.20	144	60c	70c

$1.30

7 10c stamps

and

5 12c stamps

In response to the query, "could you explain what you were doing?", he replies:

> "I just had a look at what the total amount was for each of them and then I picked a middle figure for the 12 cent ones and came up with the price for that ... got lucky the first time."

His use of paper and conventional notation is weak. He uses the letter x to represent an unknown but never uses it. He guessed 5 for the number of 12 cent stamps, multiplied 12 by 5, then multiplied 10 by 7, noticed that 60 and 70 added to 130 and was done. An intelligent performance that solved the problem - intuition, a sense of relative values and an ability to keep the various features of the problem clear (i.e. not confusing the number of stamps with their respective values). In many respects a sophisticated piece of mathematical reasoning - but not an algorithmic one. The following quotation appeared in a recent book by Bastick (1982, p. 2):

> "If man is to use his capabilities to the full and with the confidence that fits his powers, he has no alternative but to recognize the importance and power of intuitive methods in all fields of inquiry - literature and mathematics, poetry and linguistics" (Bruner, J. and Clinchy, B., 1966, p. 82).

The other student's written work provides even less information - essentially he simply writes down his final (incorrect) answer: 6 12 cent stamps and 7 10 cent stamps.

> "I thought that there was a number ending in nought ... that the 12 cent stamps had to come from a number ending in nought, so 5 did that (he wrote 6 down on the paper - otherwise he would have had the correct answer) so 60 ... had 70 cents left, so 7 of them."

A similar approach to the first student - a fairly sophisticated sensitivity to relative magnitudes and patterns (eg. keying on the zeroes), virtually no intermediate written work and no recourse to conventional algorithmic approaches. Very similar comments apply to both of the middle ability students, both of whom arrive at the correct answer by a process of guessing and noticing the low-order zeroes in some of the figures.

One of the low ability students also used a similar approach but although he obtains the correct answer he lacks confidence in his solution:

> "...That's what she could have bought... either that or something different".

It is not clear from this whether he thinks his answer could be wrong or whether he thinks that the problem could have more than one correct answer. A more perceptive interviewer might have been able to clarify this. The remaining low ability student also appears to lack confidence.

> "I have never done anything like this since about the second form [3 years ago] ... I don't really know."

She tried dividing 10 into 130, getting 13 and then divided 12 into 130 but gave up when it didn't come out evenly. She seems to be relying on a rule which she can't remember (the rote approach to mathematics) rather than on an understanding of either the situation or on an understanding of the various relationships among the numbers.

Although any conclusions based on 6 responses to one question are premature, a couple of comments deserve mention. First, it is noteworthy that all 6 students across the ability range all select variations of the same strategy - a heuristic form of guessing. In the absence of appropriate clues - such as, 'this is the next class in our unit on simultaneous equations' - not one student recognized the problem as being ameniable to such an approach. Finally, and perhaps controversially, this could be taken as a sign of strength rather than weakness. In spite of our attempts to teach appropriate algorithms, the students' natural tendency is to resort to a meaningful strategy - one they understand - rather than to a procedure that may have limited meaning, even when they can apply it properly.

<u>Problem 2</u>

$$x + 5y = 14$$
$$2x + y = 10$$

This is a classic 2 equations in 2 unknown algebra problem. The problem presents no difficulty for either of the high ability students nor for one of the average ability students. Their written work is excellent - each step is appropriately written out and executed.

$$2x + 10y = 28$$
$$2x + y = 10$$
$$9y = 18$$
$$y = 2.$$

$$2x + 2 = 10$$
$$2x = 8$$
$$x = 4.$$

The interviewer asked one student, "When you first started this question you multiplied the first equation through by 2. Why did you do that?"

> "You see you've got 1 x there and 2x there, so if you multiply that by 2 then you don't have to do anything to that one to subtract the x's so as you've only got the one lot of y's"

The other average ability student ran into a bit of difficulty.

$$-31 + 45 = 14$$

$$x + 5y = 14$$

$$2x + y = 10$$

$$-2x + 10y = -28$$

$$2x + y = 10$$

$$-9y = -18$$

$$y = 9$$

$$x + 45 = 14$$

$$x = -31$$

$$\begin{array}{r} 45 \\ -31 \\ \hline =14 \end{array}$$

After multiplying the first equation by -2, he adds the two equations, obtaining:

$$-9y = -18$$

However, from this he obtains $y = 9$. This may be a careless writing error: he is thinking 2 times 9 is 18 but writes down the 9 instead of the 2. Is this lack of attention to detail the stumbling block that differentiates the high ability students from the average ability students in our school systems ? He correctly substitutes this (incorrect) value back into the first equation to get a value for x. He then substitutes both values into the first equation which confirms (to him) that it checks. He is not aware of the logical error in verifying his answer on the wrong equation. This would appear to be an excellent opportunity for "intelligent intervention" on the part of the teacher. Yet for such intervention to occur a necessary prerequisite is for the teacher to be with the student when this particular error manifested itself. Not a likely event in most classrooms. One suggestion would be to encourage teachers to spend a greater proportion of their time watching students doing their work (and concomitantly, to spend less time teaching to the whole class). Another possibility is to design some intelligent instructional programs for microcomputers which could provide some form of appropriate message when various types of errors are made. Hayes-Roth, Waterman & Lenat (1983), Sime and Coombs (1983), Sleeman and Brown (1982) and Walker and Hess (1984) have all edited books that provide many ideas along these lines. They are all highly recommended for those who find the topic appealing.

The two low ability students both begin promisingly, but both make errors and one gets himself into an intractable situation. How often does it happen that the weaker students, because of their mistakes, end up with much more difficult situations than the stronger students? What should we be doing in mathematics education to minimize this? The first student obtains the equation

$$28 - 10y + y = 10$$

and then writes

$$28 - 11y = 10$$

This leads to $y = 1\ 7/11$. She scratches this out and goes back over her work, realizes that $-10y + y$ gives $-9y$ and then proceeds without further difficulty. Suspecting the reason, the interviewer asks: "When you originally got $y = 1\ 7/11$, why did you not go right on and substitute for x at that point?".

> "Oh, well, because they usually turn out whole numbers instead of fractional numbers and I often make that mistake doing equations."

Suspicion confirmed. This principle is rarely, if ever, explicitly taught, yet many students seem aware of it. Furthermore, the principle is a relic of the pre-calculator/computer age when problems were constructed to reduce computational effort.

The other boy's written work indicates that he begins strongly, correctly solving for y.

$$\cancel{2x} + 10y = 28$$
$$\cancel{2x} + y = 10$$
$$9y = 18$$
$$y = 2$$

$$x + 10y = 14$$
$$2x + y = 10$$
$$\cancel{2x} + 20y = 28$$
$$\cancel{2x} + y = 10$$
$$19y = 18$$
$$y = \frac{18}{19}$$

After obtaining y = 2, he makes a mistake by both substituting and then not substituting for y in the first equation. He began by rewriting the two equations:

$$x + 10 = 14$$

and

$$2x + y = 10$$

However he then inserted a y after the 10 in the first equation - presumably because he felt he should still have two equations in two unknowns. He fails to see the significance of the equation x + 10 = 14. He proceeds with his two new equations, once again solving correctly for y, this time obtaining y = 18/19. He now has two different values for y and becomes confused. Once again, this episode would seem to provide an opportunity for intelligent intervention.

From a cognitive perspective, such intervention should attempt to focus on a meaningful statement such as "Once you have a value for one unknown, the objective then becomes that of finding a value for the other", as well as on an algorithmic approach, "Substitute the value of y into the first equation". We should emphasize the 'why' as well as the 'how'.

Two comments before diccussing the third problem. First, all of the students are reasonably facile with algebraic manipulation and with approaches to solving two simultaneous equations. Second, this is in marked contrast to their approach with the first question where not one student utilized such an approach. Minsky's (1975) notion of frames and Schank and Abelson's (1975) notion of scripts (Abelson, 1981) both seem to provide relevant perspectives for this phenomenon. The third question is particularly interesting because it is similar in nature to the first question, except now the students have been exposed to question 2.

Problem 3

> A fixed charge is made for each car carried across a ferry, and an additional charge is made for each person carried. If 10 cars containing 30 people are carried, \$9.50 is collected. If 12 cars containing 40 people are carried, \$12.00 is collected. What is the cost for each car and the cost for each person?

In contrast to the previous question, not one student was able to successfully solve this problem. One of the high ability students began by correctly forming two equations, although he uses words instead of letters for the variables:

10 cars + 30 people = \$9.50

12 cars + 40 people = \$12.00

He then divides both equations by 2. This is a clear divergence from the algorithm he just finished using in the previous problem. Instead it is more like a reversion to the heuristic approach he used in the first problem, even though he has the surface structure of two equations (although they are not represented in strict algebraic form). It is difficult to conjecture what the appropriate features are that activate one "script" or "frame" instead of another. The apparent rationale is to try and get smaller numbers, perhaps so one can spot an obvious pattern or relationship.

5 cars + 15 people = \$4.75

6 cars + 20 people = \$6.00

At this point he realizes that he can eliminate the cars variable by successive multiplication by 6 and 5 and then subtract. He obtains

30 cars + 90 p = 28.50

30 cars + 100 p = 36.00

The second equation is incorrect - even strong students make simple errors in arithmetic. This may be gratifying news to some of the weaker students, but it is a frustrating fact for teachers, researchers and, sometimes, for the student. In this case the error fails to become troublesome because a new difficulty arises. Obtaining a value for p of 75 cents, he then loses sight of the problem and fails to substitute back.

> "I didn't do it right. Oh well ... You can't do it that way can you ? Two cars contain 30 ... oh yeh, I see what I did wrong ... can't do it that way ..."
> "I was just trying to um sorta just picking the prices of the cars and finding the differences then you get a price for the car and multiply it for each one and see what

He continued to spend about four minutes trying to find an appropriate number that would work, essentially a guessing procedure (undoubtedly non-random), with no success.

The other high ability boy also begins with a logical guessing approach. He writes:

30 x 5 = 150

and 12 x .80 = 960

> "so if every person was charged 5 cents, it would get rid of the 50 cents. Therefore every car cost 80 cents." "I just started off saying if it was thirty people go across there ... to get that back to an even number again, to get it back to even dollars. I just said 30 times 5, 30 people $1.50 ... so if the car was ... so they'd make them 5 cents each ... there's only 8 dollars left so you are going to get 80 cents a car, because there's only 10 cars ... and I tried it again for the $12. and it doesn't work!"

The logic of his approach is sound. The strategy is also intelligent - look for certain patterns among the numbers (such as working with integral values) and try to find a set of numbers that fit the pattern. Thus he tries to find values that get rid of the 50 cents so he can work with just dollars. The student should receive recognition for the reasonableness of his approach. Unfortunately this approach fails to leave much of an audit trail for the teacher and would have to remain unrecognized unless the teacher was beside the student at the time.

> "so I don't know a thing from that, so I'll do it all over again ... there must be an easier way ...it's just dawned on me how dumb I am!"

He begins writing two equations.

> E: "Why do you all of a sudden go to two equations ?"

S: "It just dawned on me that there's 2 sets of figures and you had 2 unknowns ... there's only 2 unknowns and there's 2 ways that they are connected to different numbers ... that this [2 equations] was an easier way to figure it out than that [selective guessing]"

He then writes

$$30x + 10y = 9.50$$

$$40x + 12y = 12.00$$

and then (!)

$$70x + 40y = 28.5$$

$$70x + 36y = 36$$

$$4y = -12.5$$

"How can you get a negative cost? Done something wrong."

This appears to be a curious mixture of multiplying by 3 and adding 30 or 40. Why is it that a strong student can correctly manipulate the algebra one moment (ie. in the previous problem) and make such a mess of it the next? This would appear to be one fact that our theories are going to have to account for. He recognizes the inappropriateness of his answer, but has now spent almost 9 minutes on this question and is quite willing to move on to another problem.

The remaining four students all fail to produce any written work for this problem. Comments from the two middle ability students were:

S: "Can I miss this one?"
E: "Sure. It's a beaut of a question isn't it?"
S: "Yeh."
E: "What were you thinking when you were looking at it?"
S: "Well, um, I thought perhaps if you added the cars

together and the cost and sort of then worked that out per car and then the people added them; together and worked it out like that ... I can't sort of figure out these. I gotta try and divide everything into such and such."

One statement was elicited from one of the low ability students:

"I haven't worked very much with those kind of things."

The problem is clearly too difficult for them to even make an attempt at. Perhaps it is a function of the verbal complexity of the question, but these latter four students appear to see no connection between this problem and algebra. Rather it appears to present them with a very difficult problem in logic, one that simply overwhelms them.

This concludes the section of three questions related to simultaneous equations. From a research perspective the data appears to be very rich - both the written work and the elicited comments provide insights into how students actually do such problems. There are implications for researchers, classroom teachers and teacher educators eminating from the analysis of the protocols. Much of the value comes from the relationships among the three problems. This is much more illuminating than just having one question from this domain. A major recommendation for further research is to pursue this idea more fully and to focus on a larger set of such problems over more than one day in order to more fully explore the stability of individual student approaches. This recommendation is consistent with much previous work that has focused on domains such as ability to work with fractions or the the child's conception of number. It might also prove interesting to share such results with the students themselves, both to give individual students less sense of being alone when they make mistakes or when they are confused as well as to give them a meta-perspective on their approaches. Perhaps this idea could be carried further, having the curriculum contain components where students learn to analyze other students' work.

The remaining 10 questions were multiple choice items taken from the ACER Mathematics Item Bank for grade 10. These questions failed to elicit any written work other than the circling of their response, even though they had in excess of 2/3 a sheet of blank paper below each item. Thus these items, inadvertently, became tests of "internal mental processes". In a research study concerned with externally manifested processes this is a severe limitation. However it is also possible to reflect on some possible implications for mathematics education. Such exams, which are a fairly common feature in many mathematics classrooms, regardless of country, permit one to make general statements about the level of competence achieved by either an individual or a group of students (classroom, school, district, or even country). These statements are usually expressed as percentages with respect to some identified domain (the class average was 72% on the Grade 10 final exam). Even when the statement is about an individual it rarely provides much information about the student's mathematical proclivities. That is a lot to ask of any number. But the real weakness of such items may be in the implicit message they provide - mathematics consists of being able to select one correct alternative from among four alternatives. It is a clear recognition that the answer is more important than the process by which it was arrived. The lack of a requirement to explicitly justify one's answer seems to imply that one should be able to do these type of problems in one's head. Certainly all six of the students in this study behaved in that fashion. This topic warrants a more detailed examination of the effect of our present evaluation procedures on the mathematics curriculum.

The author of a paper such as this faces a dilemma which becomes particularly severe at this point. Should one organize the data and the discussion by individual student or by item type? Until now the organization has been by item type, thereby permitting a contrast of different approaches. Such a format facilitates emphasizing the diversity of perspectives present in any classroom. The other main possibility is to organize the material by student, constructing an individual profile which may reveal patterns and inconsistencies, revealing the richness of human complexity. This latter approach would appear to be the more appropriate procedure if one is interested in a theory of cognitive development. The former is more appropriate if one wishes to make statements about educational practice. Although it is possible to weave a rich tapestry with these two themes, one is likely to assume figure status and the other will constitute the ground. In this particular case the data for an individual student is not so comprehensive (one 1-hour session) to justify organizing about the individual, although it seems fairly clear that many such single subject studies should be conducted which provide detailed focus on the activities of a student while engaged in school related assignments.

Problem 4

> A bucket contains 8 blue marbles, 5 green marbles, and 3 red marbles. The probability of choosing at random one blue marble, is
>
> A $\frac{1}{2}$ C $\frac{3}{16}$
>
> B $\frac{5}{16}$ D $\frac{1}{8}$

All of the students were able to successfully answer this item correctly. Five of the students, when asked, said something similar to:

> "... you add them all up ... gives you 16 ... put the 8 over the 16"

However one of the low ability students replied:

> " 'cause there's more marbles in the bucket than any other marble, and, well I just thought there would be more chances of picking out a blue marble than another color."

No recourse to probability in an arithmetic sense, but a general intuition that notices that there are more blue marbles than any other color, therefore pick the largest fraction. The logic is only partially sound - it works for this case but only because the largest alternative turns out to be the correct answer. Nonetheless a procedure that is in need of "intelligent intervention" is interpreted as correct and inadvertently provides intermittent reinforcement for the approach. To what degree is this a feature of our present school systems?

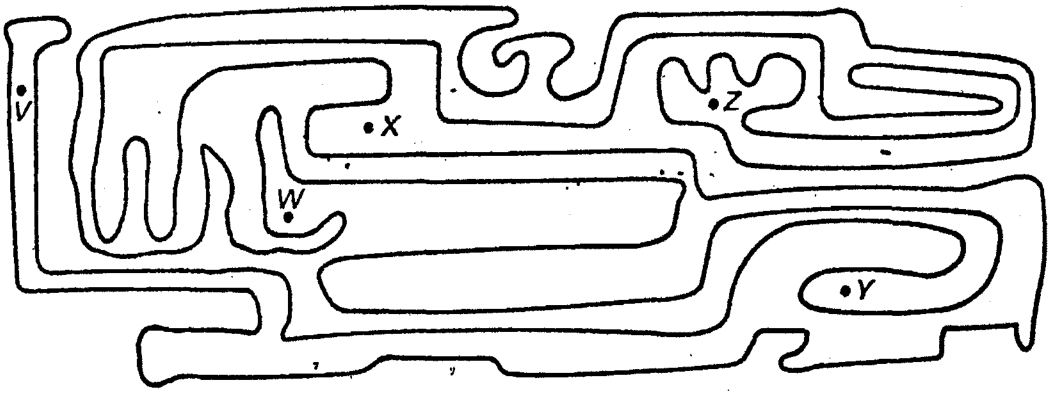
In response to a question about whether they had seen questions like that before, the responses were varied:

> "We haven't gone into any great detail or anything."
> "I can't really remember."
> "Yes. Probability and ratio and stuff like that. They did it last year."
> "Oh, nothing like that."
> "Yes, last year."

Memory plays a critical role in the development of understanding. Yet there is evidence of considerable variability in the students' ability to recall whether they had taken this topic before. This feature is important both for a theory of cognitive development and for classroom practice.

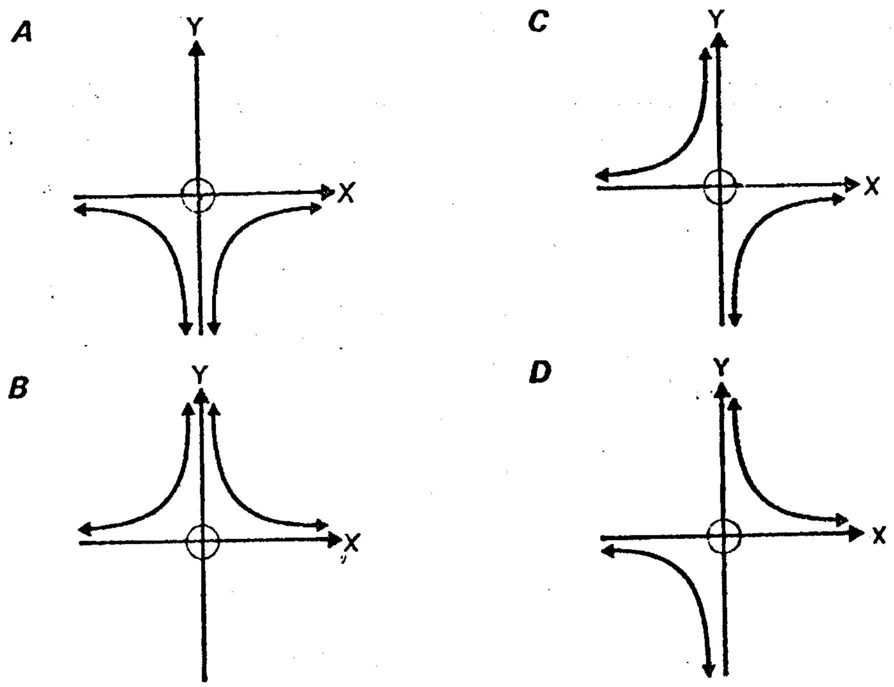
Problem 5

V is a point inside the closed curve shown. Which one of W, X, Y, and Z is also inside the curve?



Student responses to this item are marginally interesting for one reason. They all had no difficulty in selecting the correct response, and they all used the same strategy - but not one from the curriculum, which assumes they will utilize a theorem from topology involving whether or not a straight line drawn from an interior point to the outside of the figure crosses an odd or even number of lines. Instead they all treated the item like a maze, and traced their way from the one interior point until they found another interior point. It is salutary to realize that students often resort to various intelligent approaches which solve the problem, but which do not conform to the latest chapter on that topic in the curriculum. And clearly we cannot infer from a correct response that they necessarily are using the same knowledge to solve it that we assume they are.

Problem 7



Which one of the graphs could represent $y = \frac{4}{|x|}$?

A C

B D

Of the 11 multiple choice items, this item provides the most information on student processes, although even here the data is relatively scanty. One of the high ability students responded to the question of why he picked B by saying:

> "Because following an absolute value of x ... the absolute value of x is always going to be positive so therefore positive numbers are going to be a positive number, so y will always be positive."

One of the average ability students left the question when she first saw it but returned to it at the end of the session, where she essentially reiterated the above performance. In response to a query it became clear that when she saw the question the first time she failed to notice the absolute value symbol. "It just sorta didn't click." The other average student brings the difficulty out into the open.

> E: "Can you read that equation to me?"
> S: "y equals 4 over x."
> E: "Have you ever seen a number with two lines, one on each side of it?"
> S: "You mean like the absolute value of x?"
> E: "mmm"
> S: "I couldn't get it with the top bit on it. ... It was just with that joining on to that."
> E: "Right, it just looked like a box to you."
> S: "Sort of. If it was like that I would have known. ... absolute values don't have any negative numbers ... I couldn't figure out any other meaning for it so I just took it as being 4 over x".

A very slight change in the way the symbol is written makes (in this case) a large difference in how it is interpreted. For the high ability student there are enough contextual clues to ensure that the symbol refers to absolute value. But for the weaker student these same clues are not as meaningful, and in a psychological sense are not seen, thus the ambiguous symbol is not interpreted in its intended manner. However after this misunderstanding is clarified, and given that the student seems to realize that absolute values are never negative, he is still unable to solve the problem.

The two low ability students both have difficulty with this item, one exchange proceeding as follows:

> S: "[refering to the absolute value symbol] I don't know really, I just remember seeing this before
> E: "How about the phrase 'absolute value'? Is that something you have bumped into?"
> S: "I heard it before, but no."

These 11 items were very disappointing from a research perspective. They failed to elicit sufficient information - the students did not to write down any intermediate steps, and in many cases the probes to their reasons failed to generate useful comments. However for the original purpose of the study they provide a striking contrast with the information gleaned from the three simultaneous equations problems. It might be premature to write off multiple choice items entirely for collecting data on student mathematical performance, but clearly much more attention should be paid to the nature and sequencing of such items, with more careful thought given to exactly what mathematical concepts one is wishing to examine. The present attempt to sample widely from different topics within the curriculum did not provide much useful information.

Semi-structured Setting - Interview

Sample: High Ability - 1 girl, 1 boy

Average Ability - 1 girl, 1 boy

Low ability - 1 girl, 1 boy.

The following transcript has been edited to emphasize those interactions that focused on particular mathematical concepts. The session began by having the student indicate the page in her text where the class was currently working. The transcript has been divided into a sequence of three main phases, each phase dealing with a different problem or topic.

Low Ability Student

PHASE I

S: "I am not very good at maths ..."

E: "Would you be able to do this one ?"

$$\frac{3x}{5} \times \frac{5}{4x}$$

S: "I'm not really sure ... I seem to have awful trouble with my maths, I can never do it!! ... I'm not really sure where I start, we sorta haven't done that many of them."

E: "Let's try this one."

$$\frac{5}{x} + \frac{4}{x}$$

S: "We just add the 5 and the 4 which equals 9, and then we add x to the x which equals x."

E: "x and x is x ?"

S: "Yes, ... oh not usually, sometimes you could have two x. ... That could be 2x there I think [writes 9 over 2x] because [writes X x X x X = 3X] ... She then crosses out the 2, saying "They should be really x, that's right, because when you are dealing with fractions you gotta have the same denominator at the bottom.

She has two facts that she recalls, but which give her different answers. One is that x multiplied by itself 3 times is 3x, the other is that when dealing with fractions they must have the same denominator. The facts do not appear to be strictly a recall from memory but seem to be stimulated by an iconic feature - depending on what the written work looks like.

She then attempts the following question: $\frac{a}{3} + \frac{b}{4}$.

$$\frac{a}{3}+\frac{b}{4}=$$

$$\frac{4a}{12}+\frac{3b}{12}=\frac{7ab}{12}$$

E: writes: $x + \frac{5}{x} + \frac{4}{x}$

"What would this be?"

S: replaces the x with $\frac{1}{x}$

and then writes her answer $\frac{10}{x}$

E: "I just had an x and you changed it to one over x."

S: "Yes, because they are fractions and you have to have the same denominator on the bottom ... "

E: "What would happen if I wrote $\frac{x}{1} + \frac{5}{x} + \frac{4}{x}$?"

S: "you have to change the denominator, which you would have to just turn it around"

she rewrites the $\frac{x}{1}$ as $\frac{1}{x}$

This completed this episode. In many respects it reflects the growing literature on students with a weak understanding of mathematics. There is strong evidence of partial knowledge, but the procedures are not routine and there is a weak sense of when a particular rule or principle is operative. Perhaps a better image of a "frame" perspective than that of entering different rooms in a house might be that of walking down a path: a continuous model rather than a discrete one. A student such as this also presents a genuine challenge to the classroom teacher as he/she tries to determine the nature of the student's present understanding and to then build on that. Much more probing would be required to gain an appropriate picture of this student's operative mathematical knowledge, even within this context.

PHASE II

This short sequence represented another attempt to do a textbook question, in this case to simplify the expression 'square root of 8 plus the square root of 32'. Her written work is as follows:

$$\sqrt{8} + \sqrt{32} = \sqrt{2\times 4} + \sqrt{8\times 4}$$
$$2\sqrt{4} + 4\sqrt{4}$$
$$6\sqrt{4}.$$

> S: "I'm not very good at these. I am not really sure how they go. We started these at the beginning of the year. I was never very good and I seem to forget really easy how to do them."

Her written work continues to represent confusion as to the proper sequence of steps. There is no evidence of genuine understanding - it seems to be more a form of symbol manipulation.

PHASE III

In this problem, taken from near the beginning of the text, she is asked to compute (x + 3)(x + 2).

S: writes $x\ (x + 3) + (3x + 2)$.

It is almost impossible to properly appreciate this result. A request for clarification (which required the interviewer to be present at the time the problem was being solved) yielded the following:

> S: "You have got the x , so 1 times that by another x, you have two x so I put the x outside the brackets" [this explains the leading x term] ... she then rewrote the (x + 3) term ... then multiplied the 3 from the first term by the x from the second term to get 3x ... and she finally copied the 2 from the original question.
>
> S: then writes $x^2 + 3x + 3x + 9$

The first three terms follow from her previous step. However the 9 was obtained by multiplying the 3 from the (x + 3) term by the 3 from the (3x + 2) term.

Finally she writes:

$$x^2 + 9x + 9$$

She added the two coefficients for the x terms, but obtained 9 instead of 6. Mercifully, this item was now finished. It is important to recognize this performance as an accurate portrayal of a student with partial mathematical knowledge applying it as best she can to arrive at an answer. Students such as this exist, and possibly in much larger numbers than we would care to admit. Two questions that immediately spring to mind are: (1) Given a student exhibiting such an incomplete understanding, what are appropriate next steps for instruction, and (2) How can we minimize this type of confusion among the next generation of students?

This item was followed with two brief probes.

> E: "...this person multiplied y times y and said the answer was 2 y."

$$y \times y = 2y$$

> S: "That would be right."
>
> E: "... I have seen another person who uses w's, and this person went w times w and said that was w squared."

$$w \times w = w^2$$

> S: "No, it's not right because ... We've really been taught how y is ..."

Both probes reveal a fundamental misunderstanding of the nature of the algebraic representation. She is unfamiliar with using the letter w and is unable to see the equivalence of expressions with different variable labels. This is reinforced in the next exchange. She successfully multiplies 2/3 by 3/4 and 1 2/3 by 3/4.

> E: "It's nice to have some easy questions."
>
> S: "Yes, it's when you get the y's and x's... they always

> kind of complicate me. When faced with straight numbers I would be all right. ... If they had 2 times 3 times , you know, it would be easy, but you've got all these y things and they seem to have no meanings whatsoever. They have a meaning but you never know where to start."

This student's understanding of algebraic notation has strong parallels with the behavior of a pre-reader. In both cases they have not yet "broken the code', achieving that higher-order understanding of the nature of the game. To some extent expressions like $2x + 3$ are perceived as some form of spelling error. Some textbooks are now using non-alphabetic symbols like a box or a triangle to represent variables, thus possibly reducing the interference with what the student already knows about alphabetic constructions.

This open-ended interview has yielded much valuable information about this student. However, as indicated earlier, much more work would be needed to properly understand the nature of this student's understanding of mathematics. Nonetheless this particular approach would appear to yield much valuable diagnostic information. In fact this is likely the main way by which the classroom teacher obtains a better understanding of his/her students. It turns out that this student was already obtaining individual tutoring from her teacher. However teacher education programs usually focus on procedures for 'teaching the whole class' or 'covering the curriculum', and often fail to give much guidance on how to work closely with an individual student. This latter topic is usually considered the purvue of the guidance counsellor.

The remaining five protocols were not as rich in detail as the one that has just been described. However each interview elicited at least one sequence that provided a particular insight into the student's understanding of a specific topic. In the interest of brevity, only three such episodes will be given, two from an average student and one from a high ability student.

<u>Average Ability Student</u>

E: "If you were in maths class right now, what would you be doing?"

S: "... can't remember the name of them, there's equations ... they are two feet long ... we've got to solve them."

E: "Can you give me an example."

S: "x squared ... function x equals x squared, x plus h, minus function x over h, and they give us a value like function x equals 2 or something like that and we have to put it in, or function x equals x squared and we gottta go through them and solve using x squared."

E: "Could you do one for me."

$$f(x) = \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 = \frac{f(x+h)x^2 - f(x^2)}{h}$$

$$= \frac{f(x^3+x^2h) - f(x^2)}{x^2h}$$

$$= \frac{f(x^3+x^2h)}{x^2h} - \frac{f(x^2)}{x^2h}$$

$$=$$

S: "I can't remember how to do them now ... I just can't remember how we did it ... we only had one lesson of it and our maths teacher is a bit hard to understand sometimes ... you have got to substitute x squared for x + h ... when he does it, our maths teacher does it, he

> explains to you and gives you a couple of methods of doing it ... it's pretty easy to confuse them because he does them all, he might do one part of something else and go onto another part or another method and then he comes back and does it."

This student then proceeded to do the following three polynomial expansions without difficulty.

$$\begin{aligned}(x+5)(x+7) &= x^2+7x+5x+35 \\ &= x^2+12x+35.\end{aligned}$$

$$\begin{aligned}(3+y)(2-y) &= 6-3y+2y-y^2 \\ & \quad 6-y-y^2\end{aligned}$$

$$\begin{aligned}(x+2)^3 = (x+2)(x+2) &= (x^2+4x+4)(x+2) \\ &= x^3+2x^2+4x^2+8x+4x+8 \\ &= x^3+6x^2+12x+8.\end{aligned}$$

This example of competent performance is particularly valuable since it shows some of the real capabilities of the student. Ii is interesting to note the written error in the first step of the last problem. It has no impact since the mind has already moved to the next step and is not dependent on what was written before. This is much more likely to occur when students understand their material than if they are laboriously following a rule from one step to the next. Perhaps such 'inconsequential errors' could be taken as one index of understanding! There is a tendency to focus on the errors and misconceptions, in part because they are more interesting and novel than the standard algorithmic solution which we have seen and taught on numerous occasions. The careful researcher must ensure that he captures the strengths as well, since one of the difficult tasks that lies ahead for a theory of cognitive development is to adequately account for the pockets of competence as well as for the areas of confusion.

High Ability Student

E: "Do you find these questions to be fairly easy?"

S: "Most of them are relatively easy."

E: "Could you do one of them for me ... pick an easy one."

S: "... I don't know how to do them ... can't remember how to do them ... [she then proceeds to write the following for a minute and a half]"

$$
\begin{aligned}
3x - 4y + 6 &= 0 \\
x - 2y + 4 &= 0 \\
3x - 6y + 12 &= 0 \\
-10y - 6 &= 0 \\
-10y &= 6 \\
y &= -\frac{6}{10}
\end{aligned}
$$

$$
\begin{aligned}
x - 2y + 4 &= 0 \\
x + \frac{12}{10} + 4 &= 0 \\
x &= -4 - \frac{12}{10} \\
&= -5\frac{2}{10}
\end{aligned}
$$

$$\left(-5\frac{2}{10}, -\frac{6}{10}\right)$$

S: "That's right, that's how I do it ... it should be the right answer if I did it right."

E: "How would you normally find out if it was the right answer ?"

S: "Look at the back of the book."

E: "Let's see if you can explain it to me ..."

S: "... I just rewrote the second one ... and then I subtracted that from that, oh I did that wrong, I added them! [she then redoes the question]"

$$3x - 4y + 6 = 0$$
$$3x - 6y + 12 = 0$$
$$2y - 6 = 0$$
$$y = \frac{6}{2}$$
$$y = 3$$

$$x - 2y + 4 = 0$$
$$x - 6 + 4 = 0$$
$$x = 2.$$
$$(2, 3)$$

E: "You did that very fast. While you were writing it, you were just like greased lightening."

SUMMARY

One of the primary purposes of this study was to compare various procedures for obtaining data on students' mathematical knowledge. This study has shown the value in using either: (1) a set of inter-related items on the same topic, or (2) a semi-structured interview, also restricted to questions from the same domain. A relatively scattered collection of items from across the curriculum does not appear to provide a dense enough overview to make meaningful statements.

This study also provides many insights into the nature of student performance. Students often utilize heuristic approaches when attempting to solve school mathematics problems. There is considerable evidence among the low ability students of a fundamental lack of understanding of mathematical notational systems. This is particularly true for algegra where variables are represented by letters of the alphabet. All students indicate a strong facility with numeric operations but more specialized topics become potentially troublesome. Even strong students make simple errors while engaged in more difficult problems.

Finally, it is suggested that teachers might be advised to spend more time listening to students and to watching them solve problems. This implies having a greater appreciation of students' understanding of mathematics. The real game in schools is the Psychology of Mathematics Education.

This paper was supported by a grant from the School of Graduate Studies and Research at Queen's University.

REFERENCES

Abelson, R. P. (1981). Psychological status of the script concept. American Psychologist , 36 , 715-729.

Australian Council for Educational Research (1978). Australian Item Bank Progam. Mathematics Item Bank. Books 1 & 2. West Melbourne, Victoria: Jenkin Buxton Printers.

Bastick, T. (1982). Intuition: How we think and act . Chichester, England: John Wiley.

Burnett, J. D. (1981). Educational psychology and mathematical knowledge: An analysis of two student protocols [Special issue]. Focus on Learning Problems in Mathematics , 3 (2 & 3).

Davis, R. B. (1984). Learning mathematics . Beckingham, Kent: Croom Helm.

Ericsson, K. A. & Simon, H. A. (1984). Protocol analysis . Cambridge, Mass.: MIT Press.

Hayes-Roth, F., Waterman, D.A. & Lenat, D.B. (Eds.). (1983). Building expert systems . Reading, Mass.: Addison-Wesley.

Krutetskii, V. A. (1976). The psychology of mathematical abilities in schoolchildren . Chicago: University of Chicago Press.

Minsky, M. (1975). A framework for representing knowledge. In P. Winston (Ed.), The psychology of computer vision . New York: McGraw-Hill.

Newell, A. & Simon, H. A. (1972). Human problem solving . Englewood Cliffs, N. J.: Prentice-Hall.

Schank, R. C & Abelson, R. P. (1975). Scripts, plans and knowledge. In Advance Papers of the Fourth International Joint Conference on Artificial Intelligence (pp. 151-157). Tbilisi, Georgia, USSR.

Schon, D. A. (1983). The reflective practitioner . New York: Basic Books.

Sime, M. E. & Coombs, M.J. (Eds.). (1983). Designing for human-computer communication . London: Academic Press.

Sleeman, D. & Brown, J. S. (Eds.). (1982). Intelligent tutoring systems . London: Academic Press.

Walker, D. F. & Hess, R. D. (Eds.). (1984). Instructional software . Belmont, Calif.: Wadsworth.